

# On the evaluation of form factors and correlation functions for the integrable spin- $s$ XXZ chains via the fusion method

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## Abstract

Revising the derivation of the previous papers [1, 2, 3], for the integrable spin- $s$  XXZ chain we express any form factor in terms of a single sum over scalar products of the spin-1/2 XXZ chain. With the revised method we express the spin- $s$  XXZ correlation function of any given entry at zero temperature in terms of a single sum of multiple integrals.

## 1 Form factors for the spin- $s$ XXZ spin chains

Recently, a systematic method for evaluating the higher-spin form factors and correlation functions for the integrable spin- $s$  XXZ spin chain has been constructed through the fusion method [1, 2, 3]. However, the method was not completely correct. There was a non-trivial assumption that the monodromy matrix should commute with the projection operator at an *arbitrary* rapidity. The transfer matrix may be non-regular or even singular if the rapidity is equal to one of inhomogeneous parameters forming complete strings, so that the quantum inverse-scattering formulas do not necessarily hold there. In this note we revise the method and show a formula by which we can express any spin- $s$  form factor in terms of scalar products of the spin-1/2 operators. We also revise the multiple-integral representation of the zero-temperature correlation function of an arbitrary entry, and express it in terms of a single sum of multiple integrals.

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Let us consider the spin- $\ell/2$  representation  $V^{(\ell)}$  of the quantum group  $U_q(sl_2)$  constructed in the  $\ell$ th tensor product  $(V^{(1)})^{\otimes \ell}$  of the spin-1/2 representations  $V^{(1)}$ . The basis vectors  $||\ell, n\rangle$  derived from the highest weight vector:  $||\ell, 0\rangle = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_\ell$  are given by [1]

$$||\ell, n\rangle = \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1}^- \cdots \sigma_{i_n}^- ||\ell, 0\rangle q^{i_1 + i_2 + \cdots + i_n - n\ell + n(n-1)/2} \quad \text{for } n = 0, 1, \dots, \ell. \quad (1)$$

Let  $[n]_q$  denote the  $q$ -integer of an integer  $n$ :  $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ , and  $[n]_q!$  the  $q$ -factorial:  $[n]_q! = \prod_{k=1}^n [k]_q$ . We define the “square length” of  $||\ell, n\rangle$  by  $F(\ell, n) = (||\ell, n\rangle)^t \cdot ||\ell, n\rangle = ([\ell]_q! / [\ell - n]_q! [n]_q!) q^{-n(\ell-n)}$ . We define elementary matrices  $E^{i,j(\ell p)}$  in the principal grading by

$$E^{i,j(\ell p)} = ||\ell, i\rangle (||\ell, j\rangle)^t / F(\ell, j) \quad \text{for } i, j = 0, 1, \dots, \ell. \quad (2)$$

We now construct the spin- $\ell/2$  monodromy matrix on the spin- $\ell/2$  chain with  $N_s$  sites. Setting  $L = \ell N_s$ , we consider the  $N_s$ th tensor product  $(V^{(\ell)})^{\otimes N_s}$  in  $(V^{(1)})^{\otimes L}$ . Let us denote by  $\{w_j\}_L$  a set of arbitrary parameters  $w_j$  for  $j = 1, 2, \dots, L$ , which we call inhomogeneous parameters. We define the spin-1/2 XXZ monodromy matrix by

$$T^{(1)}(\lambda; \{w_j\}_L) = R_{0,12\dots L} = R_{0L}(\lambda - w_L) \cdots R_{01}(\lambda - w_1). \quad (3)$$

Here  $R_{jk}(\lambda_j - \lambda_k)$  denote the symmetric  $R$ -matrices acting on the  $j$ th and the  $k$ th components of the tensor product space  $(V^{(1)})^{\otimes L}$  where  $\lambda_0 = \lambda$  and  $\lambda_j = w_j$  for  $j = 1, 2, \dots, L$ . Let  $\epsilon$  be infinitesimally small. We denote by  $w_j^{(\ell; \epsilon)}$  the  $N_s$  sets of almost complete  $\ell$ -strings [2]:

$$w_j^{(\ell; \epsilon)} = \xi_b - (\beta - 1)\eta + \epsilon r_b^{(\beta)} \quad \text{for } \beta = 1, 2, \dots, \ell; b = 1, 2, \dots, N_s, \quad (4)$$

where  $r_b^{(\beta)}$  are generic. For  $\epsilon = 0$  we denote them by  $w_j^{(\ell)}$  and call them complete  $\ell$ -strings. We define  $T^{(\ell; \epsilon)}(\lambda)$  by  $T^{(\ell; \epsilon)}(\lambda) = T^{(1)}(\lambda; \{w_j^{(\ell; \epsilon)}\}_L)$  putting  $w_j = w_j^{(\ell; \epsilon)}$  for  $j = 1, 2, \dots, L$ . Let us denote by  $P_j^{(\ell)}$  the projector which maps the tensor product of the  $j$ th to the  $(j + \ell - 1)$ th components of  $(V^{(1)})^{\otimes L}$  onto the spin- $\ell/2$  representation  $V^{(\ell)}$ . We construct the spin- $\ell/2$  monodromy matrix  $T^{(\ell)}(\lambda)$  by applying the projector  $P_{1\dots L}^{(\ell)} := \prod_{b=1}^{N_s} P_{\ell(b-1)+1}^{(\ell)}$  as follows [1]

$$T^{(\ell)}(\lambda; \{\xi_b\}_{N_s}) = P_{1\dots L}^{(\ell)} T^{(\ell; 0)}(\lambda) P_{1\dots L}^{(\ell)}. \quad (5)$$

Here we have defined  $T^{(\ell; 0)}(\lambda)$  by the operator in the limit:  $T^{(\ell; 0)}(\lambda) = \lim_{\epsilon \rightarrow 0} T^{(\ell; \epsilon)}(\lambda)$ . We shall denote the  $(\epsilon, \epsilon')$ -element of the spin-1/2 monodromy matrix  $T^{(\ell; 0)}(\lambda)$  by  $T_{\epsilon, \epsilon'}^{(\ell; 0)}(\lambda)$ , such as  $T_{0,1}^{(\ell; 0)} = B^{(\ell; 0)}(\lambda)$ . We shall denote by  $\{\lambda_k\}_M$  a set of parameters  $\lambda_k$  for  $k = 1, 2, \dots, M$ .

Let  $|0\rangle$  be the vacuum:  $|0\rangle = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_L$ . We introduce variables  $\epsilon'_\alpha$  and  $\epsilon_\beta$  which take only two values 0 and 1 for  $\alpha, \beta = 1, 2, \dots, \ell$ . We define  $e_j^{\epsilon', \epsilon}$  ( $\epsilon', \epsilon = 0, 1$ ) by a two-by-two matrix which acts on the  $j$ th site with only one nonzero element 1 at the entry of  $(\epsilon', \epsilon)$  for each  $j$  with  $1 \leq j \leq \ell$ . We shall define  $\epsilon'_j, \epsilon_j$  and  $e_j^{\epsilon', \epsilon}$  also for  $j$  satisfying  $1 \leq j \leq L$ , later.

**Proposition 1.1.** *For arbitrary parameters  $\{\mu_k\}_N$  and  $\{\lambda_\gamma\}_M$  with  $i_1 - j_1 = N - M$  we have*

$$\begin{aligned} \langle 0 | \prod_{k=1}^N C^{(\ell)}(\mu_k) \cdot E_1^{i_1, j_1(\ell p)} \cdot \prod_{\gamma=1}^M B^{(\ell)}(\lambda_\gamma) | 0 \rangle &= F(\ell, i_1) / F(\ell, j_1) \cdot q^{i_1(\ell - i_1)/2 - j_1(\ell - j_1)/2} \\ &\times \sum_{\{\varepsilon_\beta\}} \langle 0 | \prod_{k=1}^N C^{(\ell; 0)}(\mu_k) \cdot e_1^{\varepsilon'_1, \varepsilon_1} \dots e_\ell^{\varepsilon'_\ell, \varepsilon_\ell} \cdot \prod_{\gamma=1}^M B^{(\ell; 0)}(\lambda_\gamma) | 0 \rangle. \end{aligned} \quad (6)$$

Here we take the sum over all sets of  $\varepsilon_\beta$  such that the number of integers  $\beta$  satisfying  $\varepsilon_\beta = 1$  and  $1 \leq \beta \leq \ell$  is given by  $j_1$ , while we take a set of  $\varepsilon'_\alpha$  such that the number of integers  $\alpha$  satisfying  $\varepsilon'_\alpha = 1$  and  $1 \leq \alpha \leq \ell$  is given by  $i_1$ . Each summand of (6) is symmetric with respect to exchange of  $\varepsilon'_\alpha$ s: the following expression is independent of any permutation  $\pi \in \mathcal{S}_\ell$

$$\langle 0 | \prod_{k=1}^N C^{(\ell; 0)}(\mu_k) \cdot e_1^{\varepsilon'_{\pi 1}, \varepsilon_1} \dots e_\ell^{\varepsilon'_{\pi \ell}, \varepsilon_\ell} \cdot \prod_{\gamma=1}^M B^{(\ell; 0)}(\lambda_\gamma) | 0 \rangle. \quad (7)$$

Here  $\mathcal{S}_n$  denotes the set of permutations of  $n$  integers,  $1, 2, \dots, n$ .

**Proposition 1.2.** *For a given set of the Bethe roots  $\{\lambda_\gamma\}_M$  we evaluate the scalar product (7) through Slavnov's formula of scalar products for the spin-1/2 operators*

$$\begin{aligned} \langle 0 | \prod_{k=1}^N C^{(\ell; 0)}(\mu_k) \cdot e_1^{\varepsilon'_1, \varepsilon_1} \dots e_\ell^{\varepsilon'_\ell, \varepsilon_\ell} \cdot \prod_{\gamma=1}^M B^{(\ell; 0)}(\lambda_\gamma) | 0 \rangle &= \phi_\ell(\{\lambda_\gamma\}; \{w_j^{(\ell)}\}_L) \times \\ &\times \lim_{\epsilon \rightarrow 0} \langle 0 | \prod_{k=1}^N C^{(\ell; \epsilon)}(\mu_k) \cdot T_{\varepsilon_1, \varepsilon'_1}^{(\ell; \epsilon)}(w_1^{(\ell; \epsilon)}) \dots T_{\varepsilon_\ell, \varepsilon'_\ell}^{(\ell; \epsilon)}(w_\ell^{(\ell; \epsilon)}) \cdot \prod_{\gamma=1}^M B^{(\ell; \epsilon)}(\lambda_\gamma(\epsilon)) | 0 \rangle, \end{aligned} \quad (8)$$

where  $\{\lambda_\gamma(\epsilon)\}_M$  satisfy the spin-1/2 Beth ansatz equations with inhomogeneous parameters given by the almost complete  $\ell$ -strings:  $w_j = w_j^{(\ell; \epsilon)}$  for  $j = 1, 2, \dots, L$ , and  $\phi_m(\{\lambda_\gamma\}; \{w_j\}_L)$  has been defined by  $\phi_m(\{\lambda_\gamma\}; \{w_j\}_L) = \prod_{j=1}^m \prod_{\gamma=1}^M b(\lambda_\gamma - w_j)$  with  $b(u) = \sinh(u) / \sinh(u + \eta)$ .

For even  $L$  we may assume that the ground state  $|\psi_g^{(\ell; 0)}\rangle = \prod_{\gamma=1}^M B^{(\ell; 0)}(\lambda_\gamma) | 0 \rangle$  has the spin inversion symmetry:  $U |\psi_g^{(\ell; 0)}\rangle = \pm |\psi_g^{(\ell; 0)}\rangle$  for  $U = \prod_{j=1}^L \sigma_j^x$ . We derive symmetry relations as follows.

$$\langle \psi_g^{(\ell; 0)} | e_1^{\varepsilon'_1, \varepsilon_1} \dots e_\ell^{\varepsilon'_\ell, \varepsilon_\ell} | \psi_g^{(\ell; 0)} \rangle = \langle \psi_g^{(\ell; 0)} | e_1^{1-\varepsilon'_1, 1-\varepsilon_1} \dots e_\ell^{1-\varepsilon'_\ell, 1-\varepsilon_\ell} | \psi_g^{(\ell; 0)} \rangle. \quad (9)$$

## 2 Spin- $s$ XXZ correlation functions in the massless regime

Let us now consider the spin- $s$  XXZ correlation functions, where  $2s$  corresponds to integer  $\ell$  of  $V^{(\ell)}$ . In the massless regime we set  $\eta = i\zeta$  with  $0 \leq \zeta < \pi$ . We assume that in the region  $0 \leq \zeta < \pi/2s$  the spin- $s$  ground state  $|\psi_g^{(2s)}\rangle$  is given by  $N_s/2$  sets of the  $2s$ -strings:

$$\lambda_a^{(\alpha)} = \mu_a - (\alpha - 1/2)\eta + \delta_a^{(\alpha)}, \quad \text{for } a = 1, 2, \dots, N_s/2 \text{ and } \alpha = 1, 2, \dots, 2s. \quad (10)$$

Here we also assume that string deviations  $\delta_a^{(\alpha)}$  are small enough when  $N_s$  is large enough. In terms of  $\lambda_a^{(\alpha)}$ , the spin- $s$  ground state associated with the principal grading is given by

$$|\psi_g^{(2s)}\rangle = \prod_{a=1}^{N_s/2} \prod_{\alpha=1}^{2s} B^{(2s)}(\lambda_a^{(\alpha)}; \{\xi_b\}_{N_s})|0\rangle. \quad (11)$$

Here we have  $M$  Bethe roots with  $M = 2s N_s/2 = sN_s$ . Recall that  $2s$  corresponds to  $\ell$  of  $V^{(\ell)}$ .

We shall now formulate the multiple-integral representations of the spin- $s$  XXZ correlation functions for the most general case in the massless region:  $0 \leq \zeta < \pi/2s$ . We define the zero-temperature correlation function for a given product of the spin- $s$  elementary matrices with principal grading  $E_1^{i_1, j_1(2s p)} \dots E_m^{i_m, j_m(2s p)}$ , which are  $(2s+1) \times (2s+1)$  matrices, by

$$F_m^{(2s)}(\{i_k, j_k\}) = \langle \psi_g^{(2s)} | \prod_{k=1}^m E_k^{i_k, j_k(2s, p)} | \psi_g^{(2s)} \rangle / \langle \psi_g^{(2s)} | \psi_g^{(2s)} \rangle. \quad (12)$$

For the  $m$ th product of the elementary matrices, we introduce sets of variables  $\varepsilon_\alpha^{[k]}'$  and  $\varepsilon_\beta^{[k]}$  ( $1 \leq k \leq m$ ) such that the number of  $\alpha$  satisfying  $\varepsilon_\alpha^{[k]}' = 1$  and  $1 \leq \alpha \leq 2s$  is given by  $i_k$  and the number of  $\beta$  satisfying  $\varepsilon_\beta^{[k]} = 1$  and  $1 \leq \beta \leq 2s$  by  $j_k$ , respectively. Here,  $\varepsilon_\alpha^{[k]}'$  and  $\varepsilon_\beta^{[k]}$  take only two values 0 and 1. We express them also by variables  $\varepsilon_j'$  and  $\varepsilon_j$  for  $j = 1, 2, \dots, 2sm$  as

$$\begin{aligned} \varepsilon_{2s(k-1)+\alpha}' &= \varepsilon_\alpha^{[k]}' \quad \text{for } \alpha = 1, 2, \dots, 2s; k = 1, 2, \dots, m, \\ \varepsilon_{2s(k-1)+\beta} &= \varepsilon_\beta^{[k]} \quad \text{for } \beta = 1, 2, \dots, 2s; k = 1, 2, \dots, m. \end{aligned} \quad (13)$$

For given sets of  $\varepsilon_j$  and  $\varepsilon_j'$  for  $j = 1, 2, \dots, 2sm$  we define  $\alpha^-$  by the set of integers  $j$  satisfying  $\varepsilon_j' = 1$  ( $1 \leq j \leq 2sm$ ) and  $\alpha^+$  by the set of integers  $j$  satisfying  $\varepsilon_j = 0$  ( $1 \leq j \leq 2sm$ ):

$$\alpha^-(\{\varepsilon_j'\}) = \{j; \varepsilon_j' = 1\}, \quad \alpha^+(\{\varepsilon_j\}) = \{j; \varepsilon_j = 0\}. \quad (14)$$

We denote by  $r$  and  $r'$  the number of elements of the set  $\alpha^-$  and  $\alpha^+$ , respectively. Due to charge conservation, we have  $r + r' = 2sm$ . Precisely, we have  $r = \sum_{k=1}^m i_k$  and  $r' = 2sm - \sum_{k=1}^m j_k$ .

For given sets  $\alpha^-$  and  $\alpha^+$ , which correspond to  $\{\varepsilon_j'\}_{2sm}$  and  $\{\varepsilon_j\}_{2sm}$ , respectively, we define integral variables  $\tilde{\lambda}_j$  for  $j \in \alpha^-$  and  $\tilde{\lambda}_j'$  for  $j \in \alpha^+$ , respectively, by the following:

$$(\tilde{\lambda}_{j_{max}}', \dots, \tilde{\lambda}_{j_{min}}', \tilde{\lambda}_{j_{min}}, \tilde{\lambda}_{j_{max}}) = (\lambda_1, \dots, \lambda_{2sm}). \quad (15)$$

We now introduce a matrix  $S = S((\lambda_j)_{2sm}; (w_j^{(2s)})_{2sm})$ . For each integer  $j$  satisfying  $1 \leq j \leq 2sm$ , we define  $\alpha(\lambda_j)$  by  $\alpha(\lambda_j) = \gamma$  with an integer  $\gamma$  satisfying  $1 \leq \gamma \leq 2s$  if  $\lambda_j$  is related to an integral variable  $\mu_j$  through  $\lambda_j = \mu_j - (\gamma - 1/2)\eta$  or if  $\lambda_j$  takes a value close to  $w_k^{(2s)}$  with  $\beta(k) = \gamma$ . Thus,  $\mu_j$  corresponds to the “string center” of  $\lambda_j$ . Here we have defined  $\beta(j)$  by  $\beta(j) = j - 2s[[ (j-1)/2s ]]$  for  $1 \leq j \leq M$ . Here  $[[x]]$  denotes the greatest integer less than or equal to  $x$ . We define the  $(j, k)$  element of the matrix  $S$  by

$$S_{j,k} = \rho(\lambda_j - w_k^{(2s)} + \eta/2) \delta(\alpha(\lambda_j), \beta(k)), \quad \text{for } j, k = 1, 2, \dots, 2sm. \quad (16)$$

Here  $\rho(\lambda)$  denotes the density of string centers [2], and  $\delta(\alpha, \beta)$  the Kronecker delta. We obtain the following multiple-integral representation:

$$\begin{aligned}
F_m^{(2s)}(\{i_k, j_k\}) &= C(\{i_k, j_k\}) \times \\
&\times \left( \int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\lambda_1 \cdots \left( \int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\lambda_{r'} \\
&\times \left( \int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\lambda_{r'+1} \cdots \left( \int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\lambda_{2sm} \\
&\times \sum_{\alpha^+(\{\epsilon_j\})} Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) \det S(\lambda_1, \dots, \lambda_{2sm}). \tag{17}
\end{aligned}$$

Here the sum of  $\alpha^+(\{\epsilon_j\})$  is taken over all sets  $\{\epsilon_j\}$  corresponding to  $\{\epsilon_\beta^{[k]}\}$  ( $1 \leq k \leq m$ ) such that the number of integers  $\beta$  satisfying  $\epsilon_\beta^{[k]} = 1$  and  $1 \leq \beta \leq 2s$  is given by  $j_k$  for each  $k$  satisfying  $1 \leq k \leq m$ .  $Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \dots, \lambda_{2sm})$  is given by

$$\begin{aligned}
Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) &= (-1)^{r'} \frac{\prod_{j \in \alpha^-(\{\epsilon'_j\})} \left( \prod_{k=1}^{j-1} \varphi(\tilde{\lambda}_j - w_k^{(2s)} + \eta) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}_j - w_k^{(2s)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(\lambda_\ell - \lambda_k + \eta + \epsilon_{\ell,k})} \\
&\times \frac{\prod_{j \in \alpha^+(\{\epsilon_j\})} \left( \prod_{k=1}^{j-1} \varphi(\tilde{\lambda}'_j - w_k^{(2s)} - \eta) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}'_j - w_k^{(2s)}) \right)}{\prod_{1 \leq k < \ell \leq 2sm} \varphi(w_k^{(2s)} - w_\ell^{(2s)})}. \tag{18}
\end{aligned}$$

In the denominator we set  $\epsilon_{k,\ell} = i\epsilon$  for  $Im(\lambda_k - \lambda_\ell) > 0$  and  $\epsilon_{k,\ell} = -i\epsilon$  for  $Im(\lambda_k - \lambda_\ell) < 0$ , where  $\epsilon$  is an infinitesimally small positive real number. The coefficient  $C(\{i_k, j_k\})$  is given by

$$C(\{i_k, j_k\}) = \prod_{k=1}^m (F(\ell, i_k)/F(\ell, j_k)) \cdot q^{i_k(\ell-i_k)/2-j_k(\ell-j_k)/2}. \tag{19}$$

In (18) we take a set  $\alpha^-(\{\epsilon'_j\})$  corresponding to  $\epsilon_\alpha^{[k]}'$  for  $k = 1, 2, \dots, m$ , where the number of integers  $\alpha$  satisfying  $\epsilon_\alpha^{[k]}' = 1$  and  $1 \leq \alpha \leq 2s$  is given by  $i_k$  for each  $k$  ( $1 \leq k \leq m$ ).

We can derive the symmetric expression for the multiple-integral representation of the spin- $s$  correlation function  $F_m^{(2s)}(\{i_k, j_k\})$  as follows.

$$\begin{aligned}
F_m^{(2s)}(\{i_k, j_k\}) &= \frac{C(\{i_k, j_k\})}{\prod_{1 \leq \alpha < \beta \leq 2s} \sinh^m(\beta - \alpha)\eta} \prod_{1 \leq k < l \leq m} \frac{\sinh^{2s}(\pi(\xi_k - \xi_l)/\zeta)}{\prod_{j=1}^{2s} \prod_{r=1}^{2s} \sinh(\xi_k - \xi_l + (r-j)\eta)} \\
&\times \sum_{\sigma \in \mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}} (\text{sgn } \sigma) \prod_{j=1}^{r'} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} d\mu_{\sigma j} \prod_{j=r'+1}^{2sm} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\mu_{\sigma j} \\
&\times \sum_{\{\epsilon_\beta^{[1]}\}} \cdots \sum_{\{\epsilon_\beta^{[m]}\}} Q'(\{\epsilon_j, \epsilon'_j\}; \lambda_{\sigma 1}, \dots, \lambda_{\sigma(2sm)}) \left( \prod_{j=1}^{2sm} \frac{\prod_{b=1}^m \prod_{\beta=1}^{2s-1} \sinh(\lambda_j - \xi_b + \beta\eta)}{\prod_{b=1}^m \cosh(\pi(\mu_j - \xi_b)/\zeta)} \right) \\
&\times \frac{i^{2sm^2}}{(2i\zeta)^{2sm}} \prod_{\gamma=1}^{2s} \prod_{1 \leq b < a \leq m} \sinh(\pi(\mu_{2s(a-1)+\gamma} - \mu_{2s(b-1)+\gamma})/\zeta). \tag{20}
\end{aligned}$$

Here  $\lambda_j$  are given by  $\lambda_j = \mu_j - (\beta(j) - 1/2)\eta$  for  $j = 1, \dots, 2sm$ , and  $(\text{sgn } \sigma)$  denotes the sign of permutation  $\sigma \in \mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}$ . We have defined  $\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}$  as follows [2]: An element  $\sigma$  of  $\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}$  gives a permutation of integers  $1, 2, \dots, 2sm$ , such that  $\sigma j$  satisfying  $\sigma j \equiv \beta \pmod{2s}$  are put in increasing order in the sequence  $(\sigma 1, \sigma 2, \dots, \sigma(2sm))$  for each integer  $\beta$  satisfying  $1 \leq \beta \leq 2s$ . In (20) we have defined  $Q'$  by multiplying  $Q$  by the demoninator in the second line of (18):  $Q'(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) = Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \dots, \lambda_{2sm}) \prod_{1 \leq k < \ell \leq 2sm} \varphi(w_k^{(2s)} - w_\ell^{(2s)})$ . The coefficient  $C(\{i_k, j_k\})$  is given by (19). The sums with respect to  $\{\varepsilon_\beta^{[k]}\}$  are taken over all sets  $\{\varepsilon_\beta^{[k]}\}$  ( $1 \leq k \leq m$ ) such that the number of integers  $\beta$  satisfying  $\varepsilon_\beta^{[k]} = 1$  and  $1 \leq \beta \leq 2s$  is given by  $j_k$  for each  $k$ . We take such a set  $\alpha^-(\{\varepsilon'_j\})$  that corresponds to sets  $\{\varepsilon_\alpha^{[k]'}\}$  for  $k = 1, 2, \dots, m$ , where the number of integers  $\alpha$  satisfying  $\varepsilon_\alpha^{[k]'} = 1$  and  $1 \leq \alpha \leq 2s$  is given by  $i_k$  for each  $k$  ( $1 \leq k \leq m$ ).

The spin-inversion symmetry (9) leads to useful relations among the expectation values of local or global operators. For an illustration, let us evaluate the one-point function in the spin-1 case with  $i_1 = j_1 = 1$ ,  $\langle E_1^{1,1(2p)} \rangle$ . Setting  $\varepsilon'_1 = 0$  and  $\varepsilon'_2 = 1$  in formula (6) we decompose the spin-1 elementary matrix in terms of the spin-1/2 elementary matrices

$$\langle \psi_g^{(2)} | E_1^{1,1(2p)} | \psi_g^{(2)} \rangle = \langle \psi_g^{(2;0)} | e_1^{0,0} e_2^{1,1} | \psi_g^{(2;0)} \rangle + \langle \psi_g^{(2;0)} | e_1^{0,1} e_2^{1,0} | \psi_g^{(2;0)} \rangle. \quad (21)$$

Through symmetry relations (7) with respect to  $\varepsilon'_\alpha$  and the spin inversion (9) we have

$$\langle \psi_g^{(2;0)} | e_1^{0,0} e_2^{1,1} | \psi_g^{(2;0)} \rangle = \langle \psi_g^{(2;0)} | e_1^{1,1} e_2^{0,0} | \psi_g^{(2;0)} \rangle = \langle \psi_g^{(2;0)} | e_1^{0,1} e_2^{1,0} | \psi_g^{(2;0)} \rangle = \langle \psi_g^{(2;0)} | e_1^{1,0} e_2^{0,1} | \psi_g^{(2;0)} \rangle, \quad (22)$$

and hence we have

$$\langle \psi_g^{(2)} | E_1^{1,1(2p)} | \psi_g^{(2)} \rangle = 2 \langle \psi_g^{(2;0)} | e_1^{0,0} e_2^{1,1} | \psi_g^{(2;0)} \rangle. \quad (23)$$

We thus obtain the double-integral representation of  $\langle E_1^{1,1(2p)} \rangle$  such as given in Ref. [2].

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